Microwave Resonators

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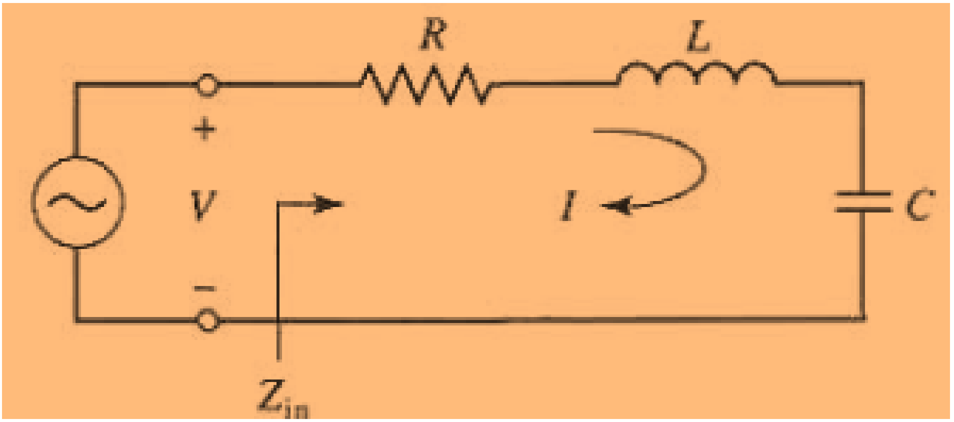
Microwave Resonators

- Applications
  - Filters
  - Oscillators
  - Frequency Meters
  - Tuned Amplifiers

- Similar to RLC resonators!

Series/Parallel Resonant Circuit



$$Z_{in} = R + j\omega L - j\frac{1}{\omega C}.$$

$$P_{in} = \frac{1}{2}VI^* = \frac{1}{2}Z_{in}|I|^2 = \frac{1}{2}Z_{in}\left|\frac{V}{Z_{in}}\right|^2 = \frac{1}{2}|I|^2\left(R + j\omega L - j\frac{1}{\omega C}\right).$$

The power dissipated by the resistor, R, is

$$P_{loss} = \frac{1}{2}|I|^2 R,$$

the average magnetic energy stored in the inductor, L, is

$$W_m = \frac{1}{4}|I|^2 L,$$

and the average electric energy stored in the capacitor, C, is

$$W_e = \frac{1}{4}|V_c|^2 C = \frac{1}{4}|I|^2\frac{1}{\omega^2 C},$$

$$P_{in} = P_{loss} + 2j\omega(W_m - W_e)$$

$$Z_{in} = \frac{2P_{in}}{|I|^2} = \frac{P_{loss} + 2j\omega(W_m - W_e)}{\frac{1}{2}|I|^2}.$$

$$Z_{in} = \frac{P_{loss}}{\frac{1}{2}|I|^2} = R,$$

$$\omega_0 = \frac{1}{\sqrt{LC}}.$$

Series/Parallel Resonant Circuit

$$Q = \omega \frac{\text{(average energy stored)}}{\text{(energy loss/second)}} = \omega \frac{W_m + W_e}{P_\ell}.$$

Q, measure of the loss of a resonant circuit!

$$Q = \omega_0 \frac{2W_m}{P_{\text{loss}}} = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 RC},$$

Behavior of the I/P Impedance at resonance:

$\omega = \omega_0 + \Delta\omega$, where $\Delta\omega$ is small

$$Z_{\text{in}} = R + j\omega L\left(1 - \frac{1}{\omega^2 LC}\right) = R + j\omega L\left(\frac{\omega^2 - \omega_0^2}{\omega^2}\right),$$

since $\omega_0^2 = 1/LC$. Now $\omega^2 - \omega_0^2 = (\omega - \omega_0)(\omega + \omega_0) = \Delta\omega(2\omega - \Delta\omega) \simeq 2\omega\Delta\omega$ for small $\Delta\omega$. Thus,

$$Z_{\text{in}} \simeq R + j2L\Delta\omega \simeq R + j\frac{2RQ\Delta\omega}{\omega_0}.$$

An useful form for distributed element resonators!

Series/Parallel Resonant Circuit

Resonator with loss can be modeled as lossless resonator whose resonant frequency has been replaced with a complex effective resonant frequency:

$$\omega_0 \longleftarrow \omega_0 \left(1 + \frac{j}{2Q}\right)$$

$$Z_{in} = j2L\left(\omega - \omega_0 - j\frac{\omega_0}{2Q}\right)$$
$$= \frac{\omega_0 L}{Q} + j2L(\omega - \omega_0) = R + j2L\Delta\omega,$$

$$Z_{in} = j2L(\omega - \omega_0).$$

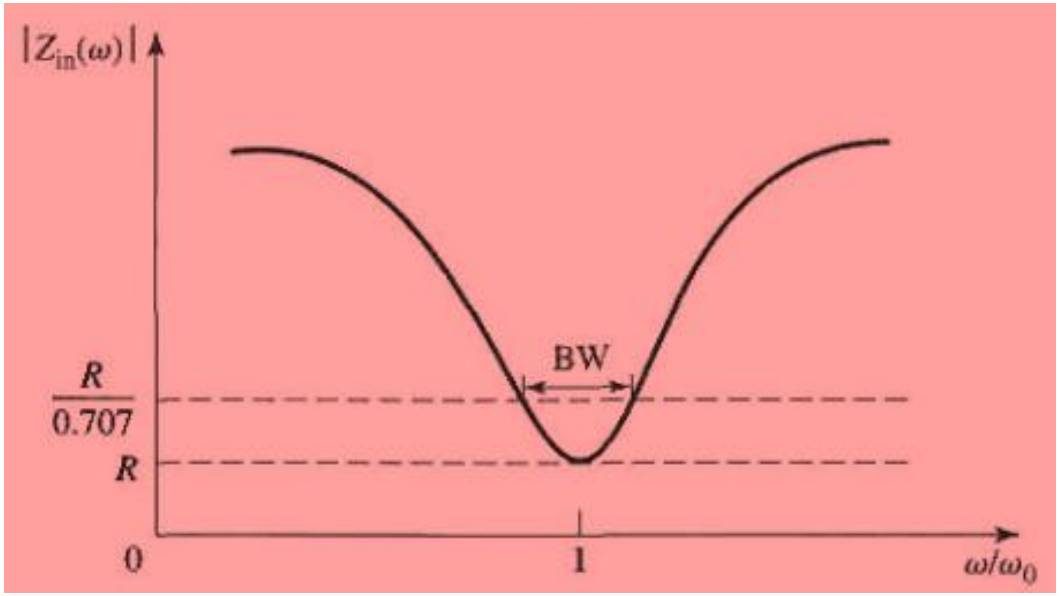
Half-Power BW:

$$|Z_{in}|^2 = 2R^2$$

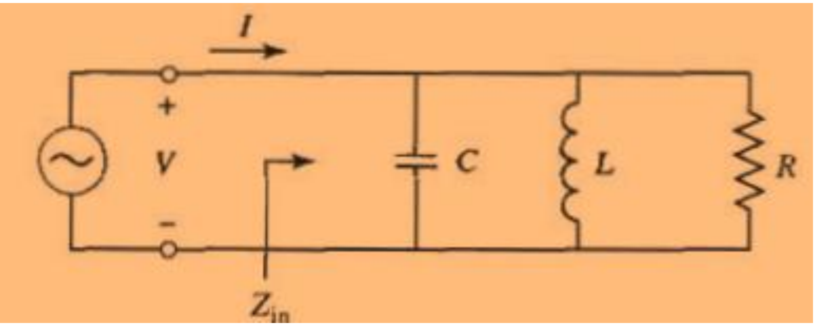
$$\Delta\omega/\omega_0 = BW/2$$

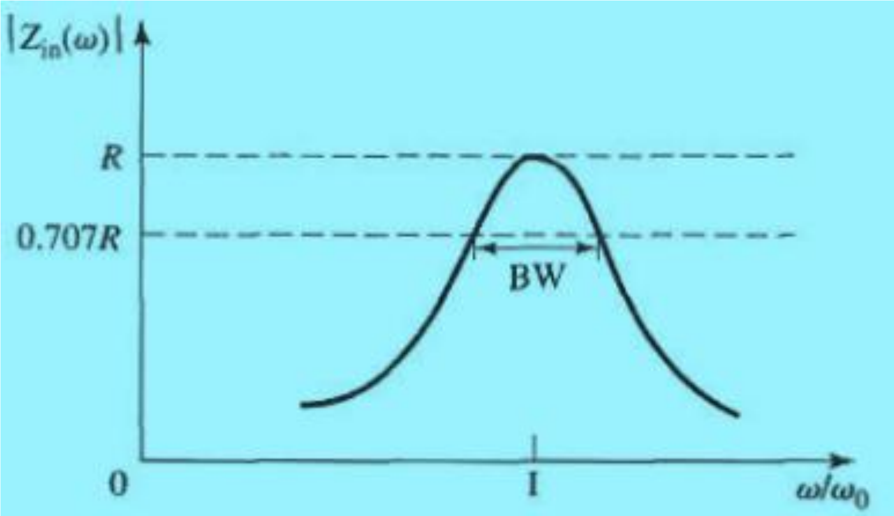
$$|R + jRQ(BW)|^2 = 2R^2$$

$$BW = \frac{1}{Q}.$$



Series/Parallel Resonant Circuit





$$Z_{\text{in}} = \left(\frac{1}{R} + \frac{1}{j\omega L} + j\omega C \right)^{-1}$$

$$P_{\text{in}} = \frac{1}{2} V I^* = \frac{1}{2} Z_{\text{in}} |I|^2 = \frac{1}{2} |V|^2 \frac{1}{Z_{\text{in}}^*}$$
$$= \frac{1}{2} |V|^2 \left(\frac{1}{R} + \frac{j}{\omega L} - j\omega C \right).$$

$$W_m = \frac{1}{4} |I_L|^2 L = \frac{1}{4} |V|^2 \frac{1}{\omega^2 L},$$

$$P_{\text{in}} = P_{\text{loss}} + 2j\omega(W_m - W_e),$$

$$P_{\text{loss}} = \frac{1}{2} \frac{|V|^2}{R},$$

$$W_e = \frac{1}{4} |V|^2 C,$$

$$Z_{\text{in}} = \frac{2P_{\text{in}}}{|I|^2} = \frac{P_{\text{loss}} + 2j\omega(W_m - W_e)}{\frac{1}{2}|I|^2},$$

$$Z_{\text{in}} = \frac{P_{\text{loss}}}{\frac{1}{2}|I|^2} = R,$$

$$\omega_0 = \frac{1}{\sqrt{LC}},$$

Series/Parallel Resonant Circuit

$$Q \simeq \omega_0 \frac{2W_m}{P_{\text{loss}}} = \frac{R}{\omega_0 L} = \omega_0 RC,$$

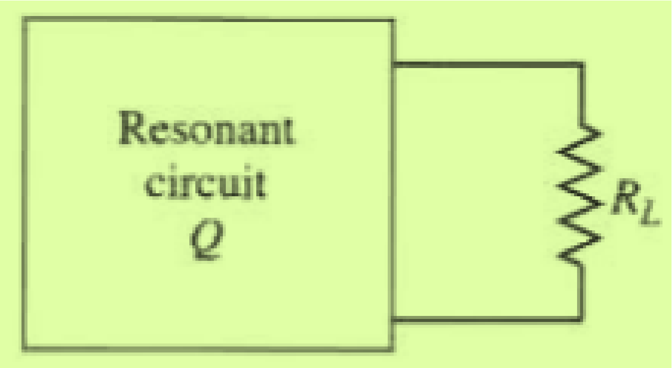
$$\frac{1}{1+x} \simeq 1 - x + \cdots.$$

Letting $\omega = \omega_0 + \Delta\omega$, where $\Delta\omega$ is small, (6.12) can be rewritten as

$$\begin{aligned}
Z_{\text{in}} &\simeq \left(\frac{1}{R} + \frac{1 - \Delta\omega/\omega_0}{j\omega_0 L} + j\omega_0 C + j\Delta\omega C \right)^{-1} \\
&\simeq \left(\frac{1}{R} + j\frac{\Delta\omega}{\omega_0^2 L} + j\Delta\omega C \right)^{-1} \\
&\simeq \left(\frac{1}{R} + 2j\Delta\omega C \right)^{-1} \\
&\simeq \frac{R}{1 + 2j\Delta\omega RC} = \frac{R}{1 + 2jQ\Delta\omega/\omega_0}.
\end{aligned}$$

Series/Parallel Resonant Circuit

- Loaded and Unloaded Q



$$Q_e = \begin{cases} \dfrac{\omega_0 L}{R_L} & \text{for series circuits} \\ \dfrac{R_L}{\omega_0 L} & \text{for parallel circuits,} \end{cases}$$

$$\frac{1}{Q_L} = \frac{1}{Q_e} + \frac{1}{Q}.$$

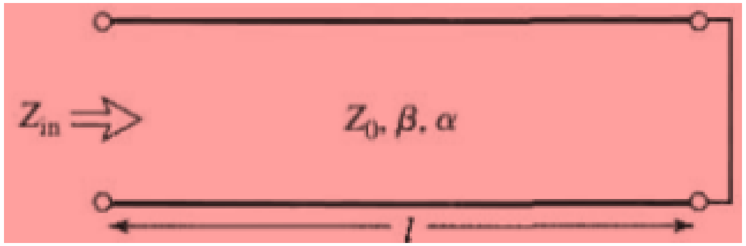
Series/Parallel Resonant Circuit

Quantity	Series Resonator	Parallel Resonator
Input Impedance/admittance	$Z_{in} = R + j\omega L - j\dfrac{1}{\omega C}$	$Y_{in} = \dfrac{1}{R} + j\omega C - j\dfrac{1}{\omega L}$
	$\simeq R + j\dfrac{2RQ\Delta\omega}{\omega_0}$	$\simeq \dfrac{1}{R} + j\dfrac{2Q\Delta\omega}{R\omega_0}$
Power loss	$P_{loss} = \dfrac{1}{2}\|I\|^2 R$	$P_{loss} = \dfrac{1}{2}\dfrac{\|V\|^2}{R}$
Stored magnetic energy	$W_m = \dfrac{1}{4}\|I\|^2 L$	$W_m = \dfrac{1}{4}\|V\|^2 \dfrac{1}{\omega^2 L}$
Stored electric energy	$W_e = \dfrac{1}{4}\|I\|^2 \dfrac{1}{\omega^2 C}$	$W_e = \dfrac{1}{4}\|V\|^2 C$
Resonant frequency	$\omega_0 = \dfrac{1}{\sqrt{LC}}$	$\omega_0 = \dfrac{1}{\sqrt{LC}}$
Unloaded Q	$Q = \dfrac{\omega_0 L}{R} = \dfrac{1}{\omega_0 RC}$	$Q = \omega_0 RC = \dfrac{R}{\omega_0 L}$
External Q	$Q_e = \dfrac{\omega_0 L}{R_L}$	$Q_e = \dfrac{R_L}{\omega_0 L}$

Transmission Line Resonators

Short-Circuited $\lambda/2$ Line

At the frequency $\omega = \omega_0$, the length of the line is $\ell = \lambda/2$,



$$Z_{in} = Z_0 \tanh(\alpha + j\beta)\ell.$$

Using an identity for the hyperbolic tangent gives

$$Z_{in} = Z_0 \frac{\tanh \alpha\ell + j \tan \beta\ell}{1 + j \tan \beta\ell \tanh \alpha\ell}$$

Observe that $Z_{in} = jZ_0 \tan \beta\ell$ if $\alpha = 0$ (no loss).

Now let $\omega = \omega_0 + \Delta\omega$,

$$\beta\ell = \frac{\omega\ell}{v_p} = \frac{\omega_0\ell}{v_p} + \frac{\Delta\omega\ell}{v_p}$$

Since $\ell = \lambda/2 = \pi v_p/\omega_0$

$$\beta\ell = \pi + \frac{\Delta\omega\pi}{\omega_0},$$

$$\tan \beta\ell = \tan\left(\pi + \frac{\Delta\omega\pi}{\omega_0}\right) = \tan \frac{\Delta\omega\pi}{\omega_0} \simeq \frac{\Delta\omega\pi}{\omega_0}.$$

$$Z_{in} \simeq Z_0 \frac{\alpha\ell + j(\Delta\omega\pi/\omega_0)}{1 + j(\Delta\omega\pi/\omega_0)\alpha\ell} \simeq Z_0\left(\alpha\ell + j\frac{\Delta\omega\pi}{\omega_0}\right),$$

since $\Delta\omega\alpha\ell/\omega_0 \ll 1$

Transmission Line Resonators

$Z_0\left(\alpha\ell + j\frac{\Delta\omega\pi}{\omega_0}\right),$

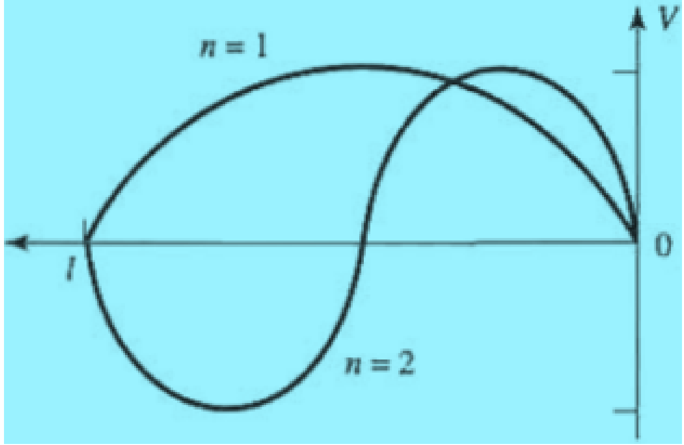
$Z_{in} = R + 2jL\Delta\omega,$

$R = Z_0\alpha\ell,$

$L = \frac{Z_0\pi}{2\omega_0}.$

$C = \frac{1}{\omega_0^2 L}.$

Resonance also occurs for $\ell = n\lambda/2, n = 1, 2, 3, \ldots.$



$$Q = \frac{\omega_0 L}{R} = \frac{\pi}{2\alpha\ell} = \frac{\beta}{2\alpha},$$

since $\beta\ell = \pi$ at the first resonance.

Transmission Line Resonators

Short-Circuited $\lambda/4$ Line

$$Z_{in} = Z_0 \tanh(\alpha + j\beta)\ell = Z_0 \frac{\tanh \alpha\ell + j \tan \beta\ell}{1 + j \tan \beta\ell \tanh \alpha\ell} = Z_0 \frac{1 - j \tanh \alpha\ell \cot \beta\ell}{\tanh \alpha\ell - j \cot \beta\ell},$$

Now assume that $\ell = \lambda/4$ at $\omega = \omega_0$, and let $\omega = \omega_0 + \Delta\omega$.

$$\beta\ell = \frac{\omega_0 \ell}{v_p} + \frac{\Delta\omega \ell}{v_p} = \frac{\pi}{2} + \frac{\pi \Delta\omega}{2\omega_0},$$

$$\cot \beta\ell = \cot\left(\frac{\pi}{2} + \frac{\pi \Delta\omega}{2\omega_0}\right) = -\tan \frac{\pi \Delta\omega}{2\omega_0} \simeq \frac{-\pi \Delta\omega}{2\omega_0}.$$

$$Z_{in} = Z_0 \frac{1 + j\alpha\ell\pi \Delta\omega/2\omega_0}{\alpha\ell + j\pi \Delta\omega/2\omega_0} \simeq \frac{Z_0}{\alpha\ell + j\pi \Delta\omega/2\omega_0},$$

since $\alpha\ell\pi \Delta\omega/2\omega_0 \ll 1$

$$Z_{in} = \frac{1}{(1/R) + 2j\Delta\omega C}. \qquad R = \frac{Z_0}{\alpha\ell} \qquad C = \frac{\pi}{4\omega_0 Z_0}. \qquad L = \frac{1}{\omega_0^2 C}$$

$$Q = \omega_0 RC = \frac{\pi}{4\alpha\ell} = \frac{\beta}{2\alpha},$$

since $\ell = \pi/2\beta$ at resonance.

Transmission Line Resonators

Open-Circuited $\lambda/2$ Line

The input impedance of an open-circuited line of length ℓ is

$$Z_{in} = Z_0 \coth(\alpha + j\beta)\ell = Z_0 \frac{1 + j \tan\beta\ell \tanh\alpha\ell}{\tanh\alpha\ell + j\tan\beta\ell}.$$

As before, assume that $\ell = \lambda/2$ at $\omega = \omega_0$, and let $\omega = \omega_0 + \Delta\omega$. Then,

$$\beta\ell = \pi + \frac{\pi\,\Delta\omega}{\omega_0},$$

and so

$$\tan\beta\ell = \tan\frac{\Delta\omega\pi}{\omega} \simeq \frac{\Delta\omega\pi}{\omega_0},$$

and $\tanh\alpha\ell \simeq \alpha\ell$. Using these results in (6.32) gives

$$Z_{in} = \frac{Z_0}{\alpha\ell + j(\Delta\omega\pi/\omega_0)}.$$

$R = \frac{Z_0}{\alpha\ell},$ $C = \frac{\pi}{2\omega_0 Z_0}.$ $L = \frac{1}{\omega_0^2 C}.$

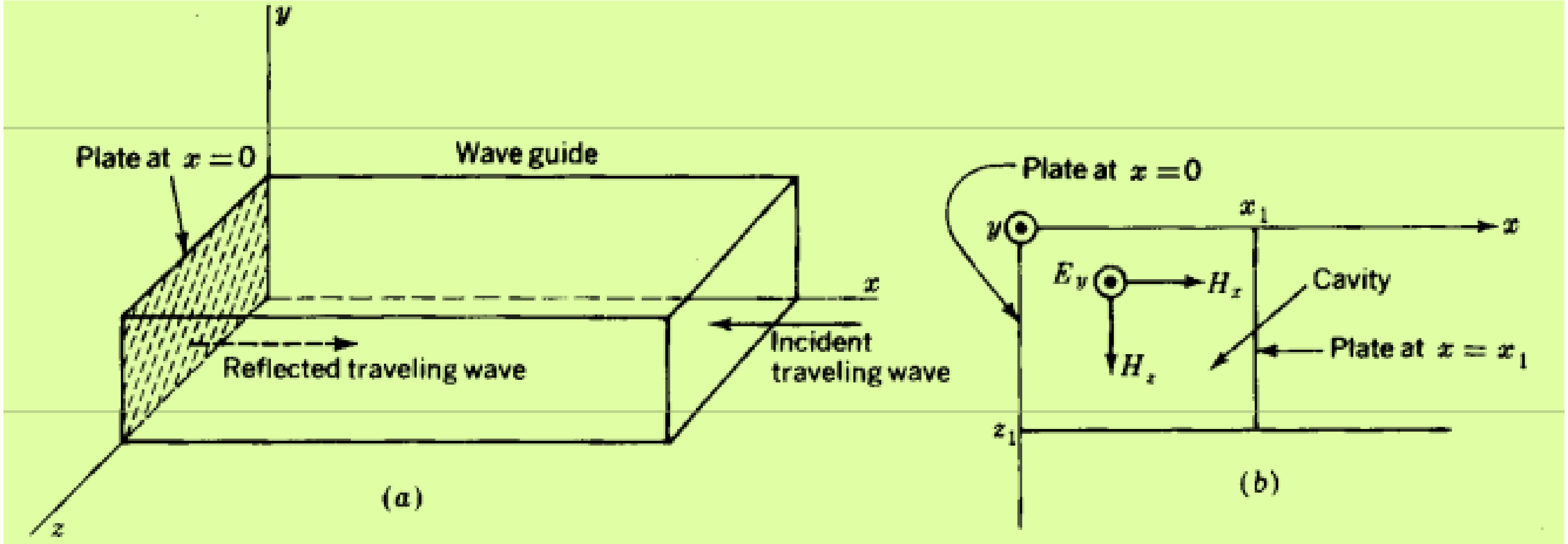
$Q = \omega_0 RC = \frac{\pi}{2\alpha\ell} = \frac{\beta}{2\alpha},$

since $\ell = \pi/\beta$ at resonance.

Rectangular Waveguide Cavities

- Because of radiation loss from open-ended waveguide, waveguide resonators are usually short circuited at both ends; closed box or cavity.
- Coupling by a small aperture, probe or loop.
- Resonant frequency and Q?

Cavity Resonator



$$E_y = \frac{j\beta Z_{yz}}{k_z} H_0 \sin k_z z\, e^{j(\omega t \pm \beta x)} \quad (1)$$

$$H_x = H_0 \cos k_z z\, e^{j(\omega t \pm \beta x)} \quad (2)$$

$$H_z = \frac{j\beta}{k_z} H_0 \sin k_z z\, e^{j(\omega t \pm \beta x)} \quad (3)$$

$$k_z = m\pi / z_1.$$

Cavity Resonator

$$E_y = \frac{-j\beta Z_{yz}}{k_z} H_0 \sin k_z z \,(e^{j\beta x} - e^{-j\beta x}) e^{j\omega t} \qquad (4)$$

$$= \frac{2\beta Z_{yz}}{k_z} H_0 \sin k_z z \sin \beta x \, e^{j\omega t} \qquad (5)$$

Inserting another conducting plate across the guide at $x = x_1$ requires that $\beta = k_x = l\pi/x_1$. Noting that the transverse-wave impedance $Z_{yz} = \omega\mu/\beta = \omega\mu/k_x$, we get

$$E_y = \frac{2\omega\mu}{k_z} H_0 \sin k_x x \sin k_z z \, e^{j\omega t} \qquad (6)$$

Proceeding in like manner for the magnetic field components, we get

$$H_x = -2H_0 \sin k_x x \cos k_z z \, e^{j[\omega t + (\pi/2)]} \qquad (7)$$

$$H_z = \frac{2k_x}{k_z} H_0 \cos k_x x \sin k_z z \, e^{j[\omega t + (\pi/2)]} \qquad (8)$$

Cavity Resonator

the designation appropriate to our example would be TE_{lm0}. Now $k^2 = k_z^2 = \gamma^2 + \omega^2\mu\epsilon$, but $\gamma^2 = -\beta^2$ ($\alpha = 0$) and $\beta = k_x$. Thus,

$$k_z^2 = -k_x^2 + \omega^2\mu\epsilon = -k_x^2 + (2\pi f)^2 \frac{1}{(f\lambda)^2}$$

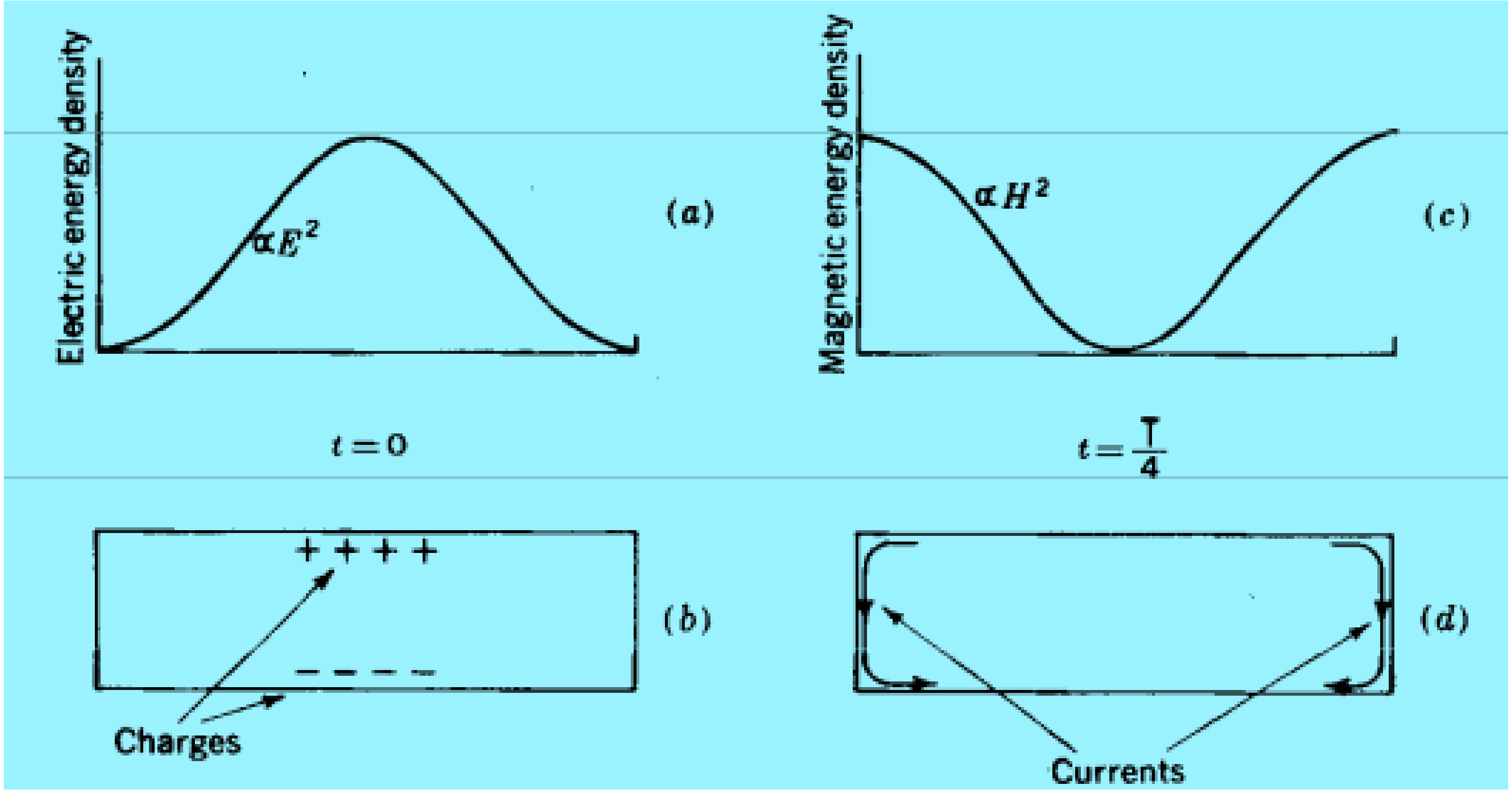
so

$$\boxed{\lambda = \frac{2}{\sqrt{(l/x_1)^2 + (m/z_1)^2}}} \qquad (9)$$

TE_{110} mode in a square-box resonator ($x_1 = z_1$) is given by

$$\lambda = \frac{2}{\sqrt{2/x_1^2}} = 1.41x_1 \qquad \text{(m)} \qquad (10)$$

Cavity Resonator



Cavity Resonator

$$Q = 2\pi \frac{\text{total energy stored}}{\text{decrease in energy in 1 cycle}}$$

or

$$Q = \frac{2}{\delta} \frac{\text{volume of cavity}}{\text{interior surface area of cavity}} \qquad (18)$$

where $\delta = 2 \text{ Re } Z_c/\omega\mu_0 = 1/e$ depth of penetration. For copper $\delta = 6.6 \times 10^{-2}/\sqrt{f}$. If $x_1 = z_1 = 100$ mm and $y_1 = 50$ mm, we have that the resonant wavelength $\lambda = 141$ mm and $Q = 17{,}500$ (dimensionless).