Waveguides and Resonators

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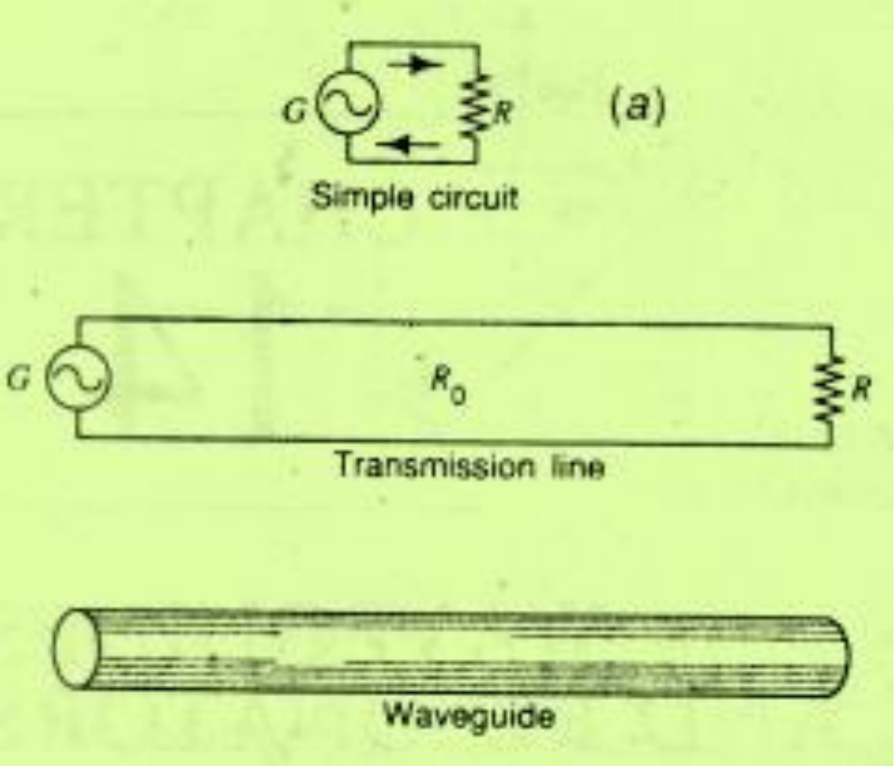
Reference Book: Electromagnetics By John D Kraus

Waveguides

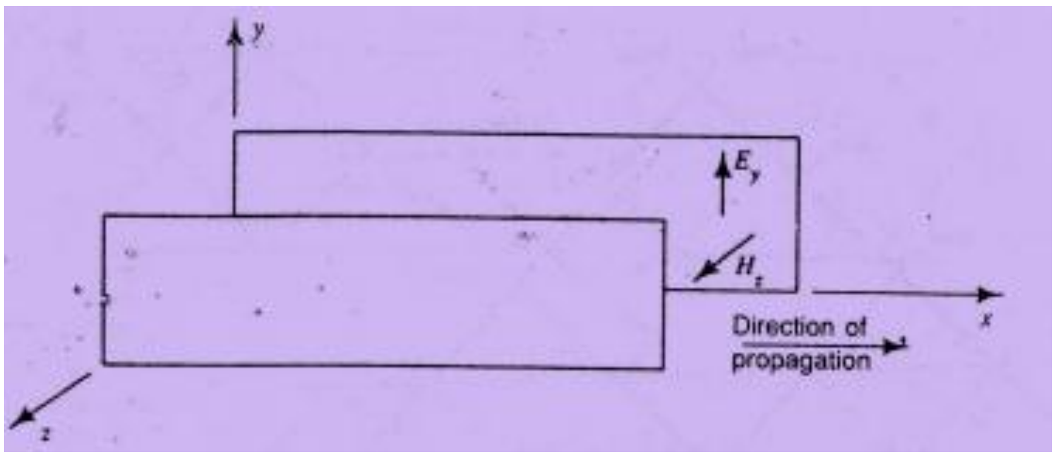
- Waveguide: Tx line which can convey EM waves only in higher-order modes.
- Circuit Theory and Field Theory



Can such a pipe convey EM energy

Waveguides

- TE Mode Wave in the infinite Parallel-Plane Tx Line or Guide



- TE: E is everywhere transverse, but H longitudinal , as well as transverse, comp.
- E_y vanishes at the sheets!

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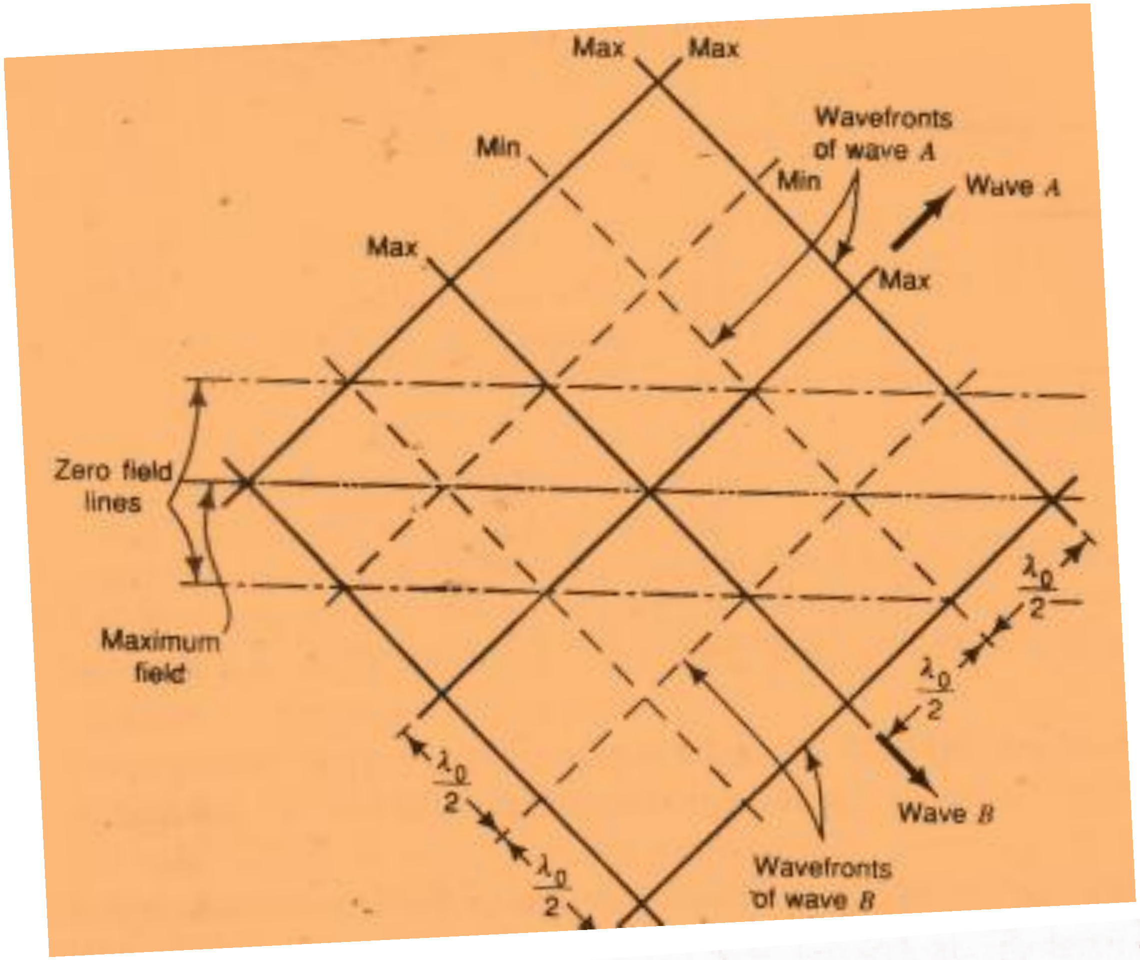

Two plane TEM waves traveling in free space in different directions result in maximum field along middle horizontal line (dash double-dot) and zero field along the two dash–dot lines.

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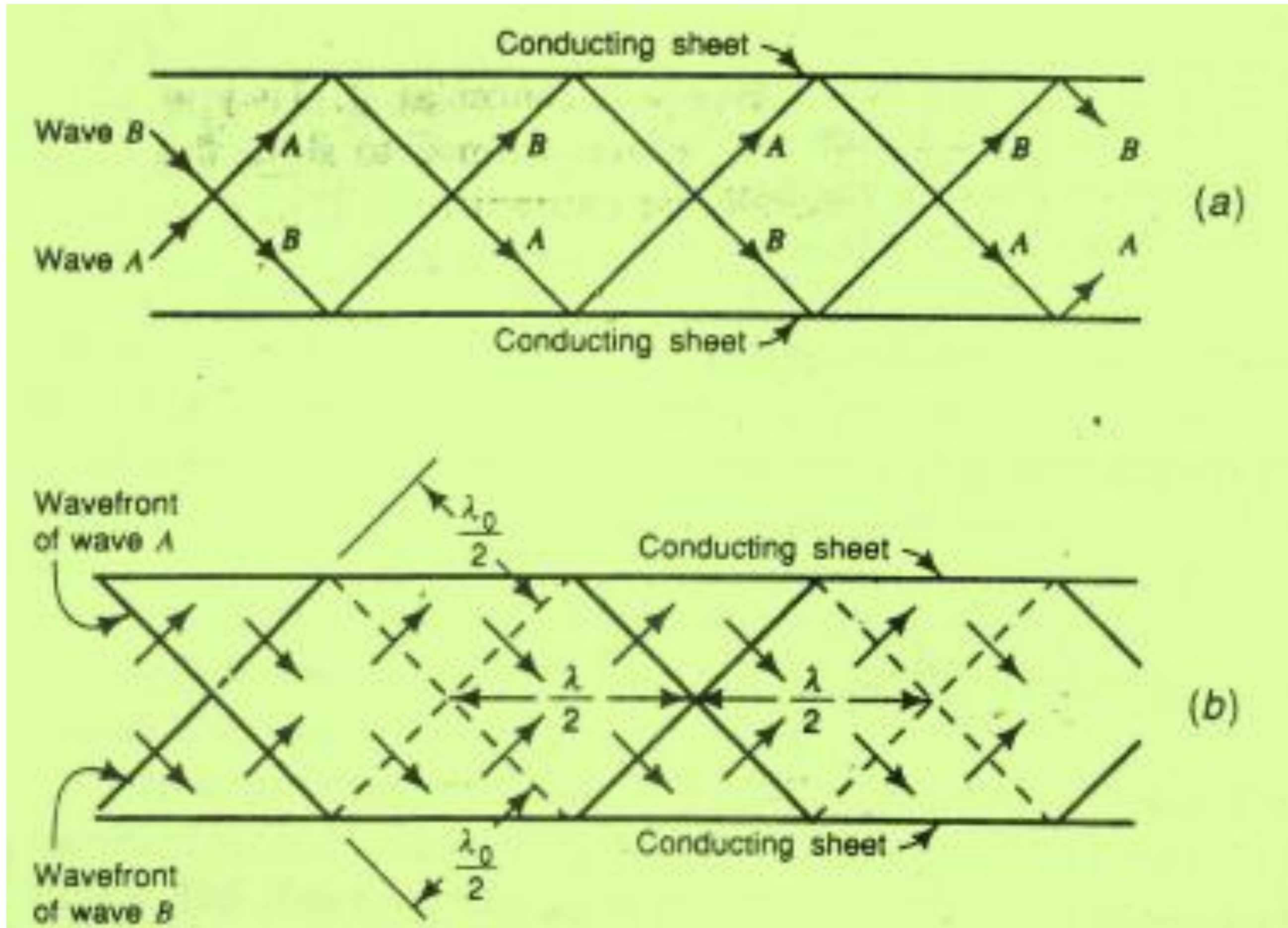
- Resultant waves belong to higher order mode.
- TE mode not transmitted unless sufficiently short wavelength: Cutoff Wavelength



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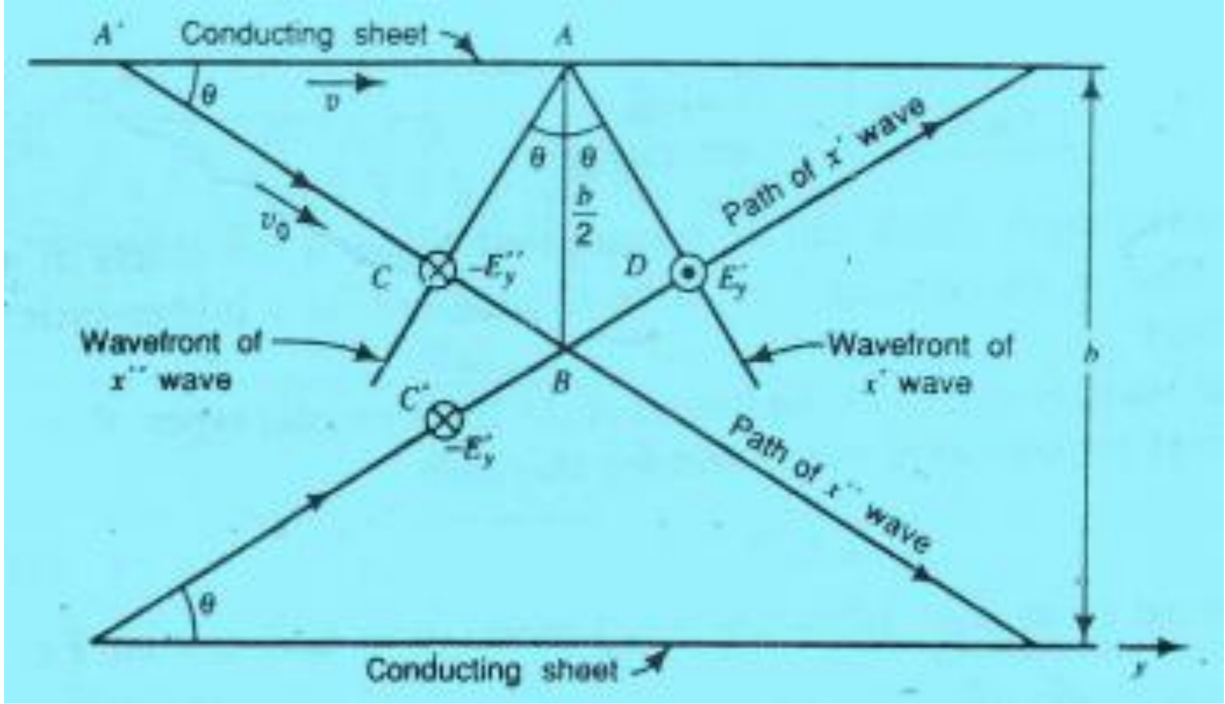


$$CB = BD = C'B = \frac{\lambda_0}{4}$$

$$CB = \frac{n\lambda_0}{4}$$

Waveguides

$$AB \sin\theta = \frac{b}{2}\sin\theta = \frac{n\lambda_0}{4}$$

$$\lambda_0 = \frac{2b}{n}\sin\theta$$

Longest wavelength:

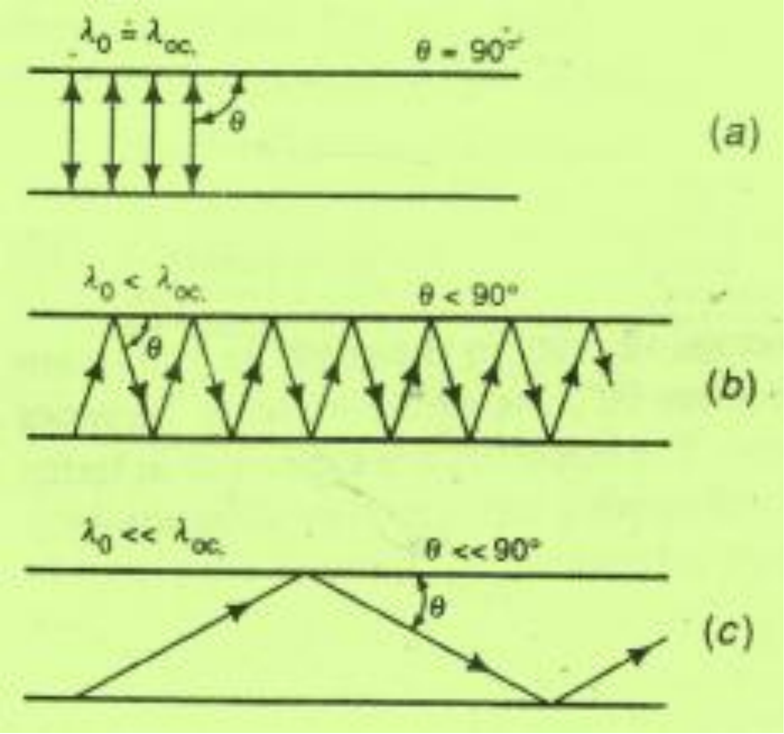
$$\lambda_{oc} = \frac{2b}{n}$$

- n corresponds to a particular higher order mode. For example, if n=1,

$$\boxed{\lambda_{oc} = 2b} \quad \text{Cutoff wavelength}$$

Waveguides

$$\theta = \sin^{-1} \frac{\lambda_0}{\lambda_{oc}}$$



Waveguides

$$v_0 = \frac{1}{\sqrt{\mu\epsilon}} \qquad (\text{m s}^{-1})$$

- Permeability (Hm^{-1}, Permittivity Fm^{-1})

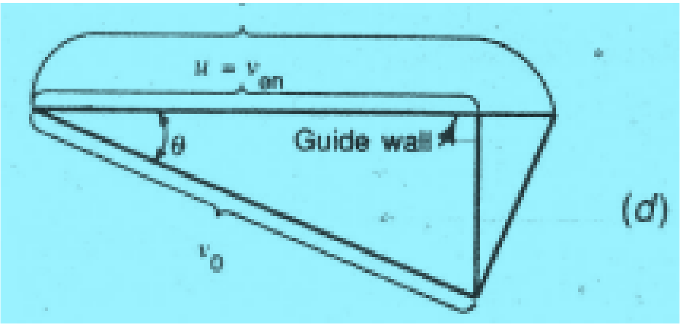
$$\frac{v_0}{v} = \frac{A'C}{A'A} = \cos\theta$$

$$v = \frac{v_0}{\cos\theta} = \frac{1}{\sqrt{\mu\epsilon}\cos\theta} \qquad (\text{m s}^{-1})$$

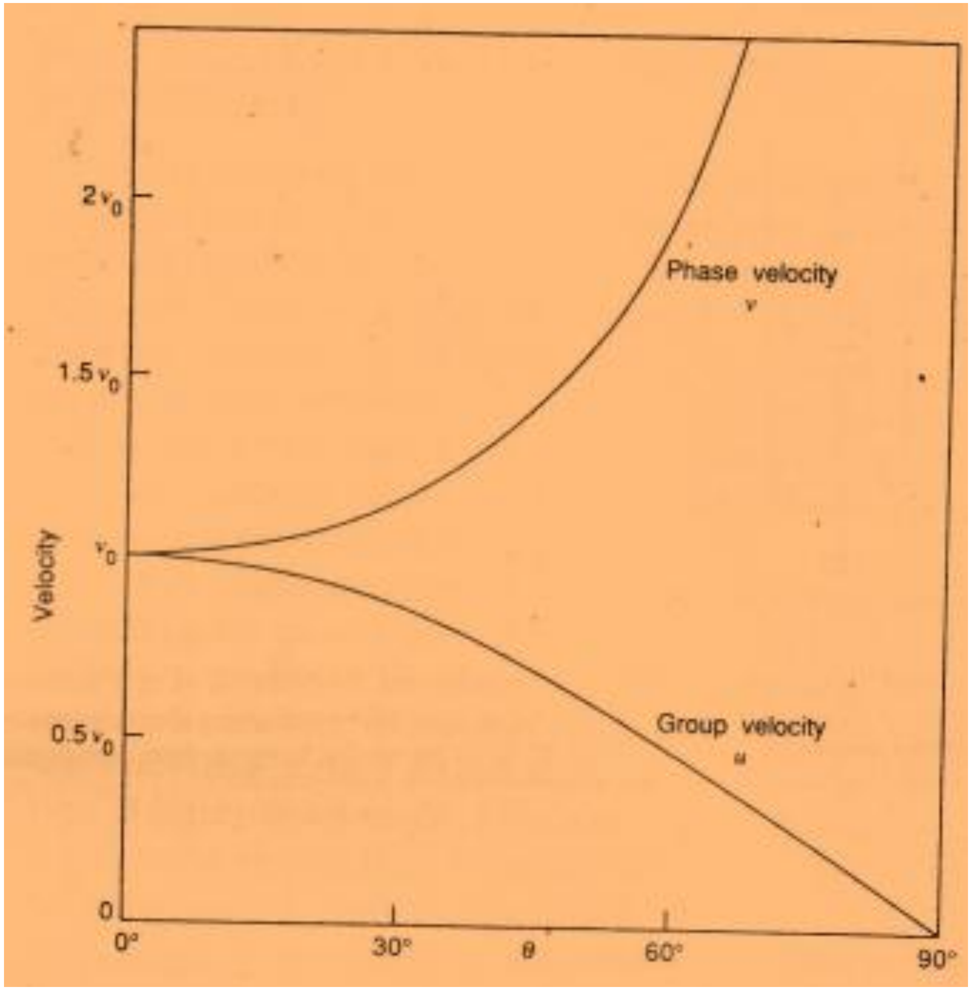
Phase velocity

- Phase velocity approaches an infinite values as the wavelength is increased towards the cutoff value. And, v approaches v_0.
- Phase velocity of higher order mode wave formed by the sheets is always equal or greater than the velocity in an unbounded medium.
- Energy velocity is always equal or less than the velocity in an unbounded medium.

Waveguides

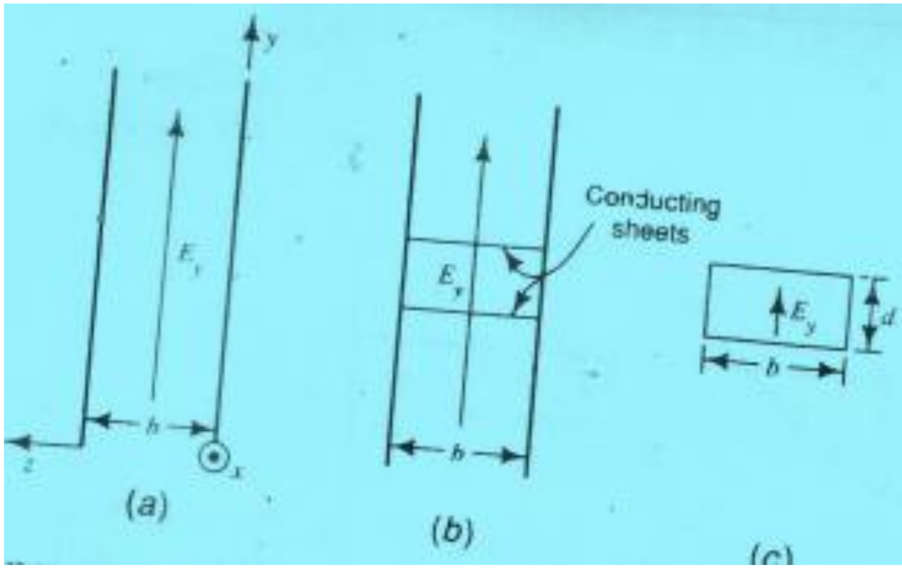


$u = v_{gn}$
θ
Guide wall
(d)
v_0



$2v_0$
$1.5v_0$
v_0
$0.5v_0$
0
Velocity
Phase velocity
v
Group velocity
u
0°
30°
θ
60°
90°

Waveguides



(*a*) Infinite-parallel-plane transmission line acting as a waveguide for TE wave. **E** is in *y* direction with wave in *x* direction (out of page). The guide consists of two parallel conducting sheets separated by a distance *b*. Additional sheets introduced normal to E_y as in (*b*) results in the hollow rectangular waveguide (*c*).

Hollow Rectangular Waveguide

- Start w. Maxwell's equation
- Restriction of harmonic variation wrt time
- Harmonic variation and attenuation wrt x
- Select mode TE or TM
- Find four fields in terms of H_x
- Develop Wave equation for H_x
- Solve this Eq. for H_x subject to boundary condtions
- Substitute H_x for other fileds

8 Steps!

Hollow Rectangular Waveguide

- Step 1

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} - \sigma E_x - \epsilon \frac{\partial E_x}{\partial t} = 0 \qquad (1)$$

$$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} - \sigma E_y - \epsilon \frac{\partial E_y}{\partial t} = 0 \qquad (2)$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} - \sigma E_z - \epsilon \frac{\partial E_z}{\partial t} = 0 \qquad (3)$$

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} + \mu \frac{\partial H_x}{\partial t} = 0 \qquad (4)$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} + \mu \frac{\partial H_y}{\partial t} = 0 \qquad (5)$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} + \mu \frac{\partial H_z}{\partial t} = 0 \qquad (6)$$

Hollow Rectangular Waveguide

- Space free from charge

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0 \qquad (7)$$

$$\frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z} = 0 \qquad (8)$$

- **Step 2-3:** Assume, field comp varies harmonically with both time and distance and also may attenuate with distance

$$E_y = E_1 e^{j\omega t - \gamma x}$$

Hollow Rectangular Waveguide

(1) to (8) reduce to

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} - (\sigma + j\omega\epsilon)E_x = 0 \tag{10}$$

$$\frac{\partial H_x}{\partial z} + \gamma H_z - (\sigma + j\omega\epsilon)E_y = 0 \tag{11}$$

$$-\gamma H_y - \frac{\partial H_x}{\partial y} - (\sigma + j\omega\epsilon)E_z = 0 \tag{12}$$

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} + j\omega\mu H_x = 0 \tag{13}$$

$$\frac{\partial E_x}{\partial z} + \gamma E_z + j\omega\mu H_y = 0 \tag{14}$$

$$-\gamma E_y - \frac{\partial E_x}{\partial y} + j\omega\mu H_z = 0 \tag{15}$$

$$-\gamma E_x + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0 \tag{16}$$

$$-\gamma H_x + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z} = 0 \tag{17}$$

$Z = -j\omega\mu \qquad (\Omega\ \mathrm{m}^{-1})$

$Y = \sigma + j\omega\epsilon \qquad (\mho\ \mathrm{m}^{-1})$

Hollow Rectangular Waveguide

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} - YE_x = 0 \tag{20}$$

$$\frac{\partial H_x}{\partial z} + \gamma H_z - YE_y = 0 \tag{21}$$

$$-\gamma H_y - \frac{\partial H_x}{\partial y} - YE_z = 0 \tag{22}$$

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} - ZH_x = 0 \tag{23}$$

$$\frac{\partial E_x}{\partial z} + \gamma E_z - ZH_y = 0 \tag{24}$$

$$-\gamma E_y - \frac{\partial E_x}{\partial y} - ZH_z = 0 \tag{25}$$

$$-\gamma E_x + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0 \tag{26}$$

$$-\gamma H_x + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z} = 0 \tag{27}$$

Hollow Rectangular Waveguide

- Step 4: TE Mode

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = 0 \tag{28}$$

$$\frac{\partial H_x}{\partial z} + \gamma H_z - YE_y = 0 \tag{29}$$

$$-\gamma H_y - \frac{\partial H_x}{\partial y} - YE_z = 0 \tag{30}$$

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} - ZH_x = 0 \tag{31}$$

$$\gamma E_z - ZH_y = 0 \tag{32}$$

$$-\gamma E_y - ZH_z = 0 \tag{33}$$

$$\frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0 \tag{34}$$

$$-\gamma H_x + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z} = 0 \tag{35}$$

Hollow Rectangular Waveguide

- Step 5:

From (32) And (33)

$$\frac{E_x}{H_y} = -\frac{E_y}{H_x} = \frac{Z}{\gamma} \qquad (\Omega) \qquad (36)$$

Transverse-wave impedance

$$Z_{yz} = \frac{E_y}{H_z} = -\frac{E_z}{H_y} = -\frac{Z}{\gamma} = \frac{j\omega\mu}{\gamma} \qquad (\Omega)$$

(37) Into (30)

$$H_y = \frac{-1}{\gamma - YZ_{yz}} \frac{\partial H_x}{\partial y} \qquad (38)$$

(37) Into (29)

$$H_z = \frac{-1}{\gamma - YZ_{yz}} \frac{\partial H_x}{\partial z} \qquad (39)$$

Hollow Rectangular Waveguide

Now, substituting (39) into (37), we obtain

$$E_y = \frac{-Z_{yz}}{\gamma - YZ_{yz}} \frac{\partial H_x}{\partial z} \tag{40}$$

and substituting (38) into (37) gives

$$E_z = \frac{Z_{yz}}{\gamma - YZ_{yz}} \frac{\partial H_x}{\partial y} \tag{41}$$

Step 6:
35:<38, 39

$$-\gamma H_x - \frac{1}{\gamma - YZ_{yz}} \left(\frac{\partial^2 H_x}{\partial y^2} + \frac{\partial^2 H_x}{\partial z^2} \right) = 0 \tag{42}$$

or

$$\frac{\partial^2 H_x}{\partial y^2} + \frac{\partial^2 H_x}{\partial z^2} + \gamma(\gamma - YZ_{yz}) H_x = 0 \tag{43}$$

Putting $k^2 = \gamma(\gamma - YZ_{yz})$ reduces (43) to

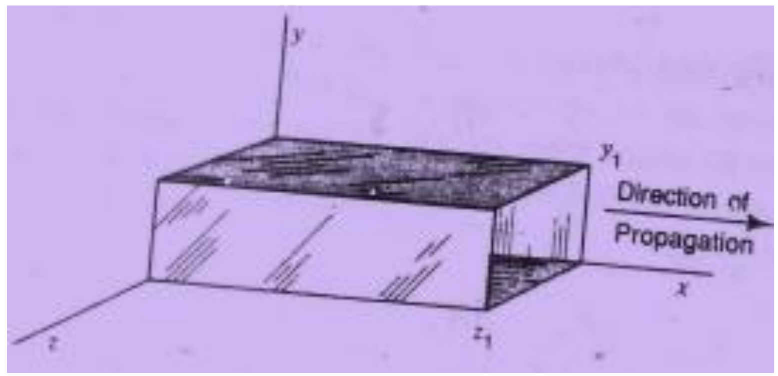
$$\frac{\partial^2 H_x}{\partial y^2} + \frac{\partial^2 H_x}{\partial z^2} + k^2 H_x = 0 \tag{44}$$

Hollow Rectangular Waveguide

- Step 7:

$$H_x = YZ \tag{45}$$

where Y = a function of y only, that is, $Y = f(y)$
Z = a function of z only



Substituting (45) in (44) gives

$$Z\frac{d^2Y}{dy^2} + Y\frac{d^2Z}{dz^2} + k^2YZ = 0 \tag{46}$$

Dividing by YZ to separate variables gives

$$\frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = -k^2 \tag{47}$$

Hollow Rectangular Waveguide

$$\frac{1}{Y}\frac{d^2Y}{dy^2} = -A_1 \quad (48)$$

$$\frac{1}{Z}\frac{d^2Z}{dz^2} = -A_2 \quad (49)$$

$$A_1 + A_2 = k^2$$

$$Y = c_1 \sin b_1 y \quad (51)$$

Substituting (51) in (48) yields

$$b_1 = \sqrt{A_1} \quad (52)$$

Hence (51) is a solution provided (52) is fulfilled. Another solution is

$$Y = c_2 \cos b_1 y \quad (53)$$

f (51) and (53) are each a solution for Y, their sum is also a solution, or

$$Y = c_1 \sin \sqrt{A_1}\, y + c_2 \cos \sqrt{A_1}\, y \quad (54)$$

Hollow Rectangular Waveguide

$$Z = c_3 \sin \sqrt{A_2}\, z + c_4 \cos \sqrt{A_2}\, z \quad (55)$$

$$H_x = c_1 c_3 \sin \sqrt{A_1}\, y \sin \sqrt{A_2}\, z + c_2 c_3 \cos \sqrt{A_1}\, y \sin \sqrt{A_2}\, z + c_1 c_4 \sin \sqrt{A_1}\, y \cos \sqrt{A_2}\, z + c_2 c_4 \cos \sqrt{A_1}\, y \cos \sqrt{A_2}\, z \quad (56)$$

Boundary conditions:
$E_y=0$ at $z=0$ and $z=z_1$
$E_z=0$ at $y=0$, and $y=y_1$

$$\sqrt{A_1} = \frac{n\pi}{y_1} \quad (57)$$

$$\sqrt{A_2} = \frac{m\pi}{z_1} \quad (58)$$

$$H_x(y, z) = H_0 \cos \frac{n\pi y}{y_1} \cos \frac{m\pi z}{z_1} \quad (59)$$

Multiplication by constant factor:

$$H_x(y, z, x, t) = H_0 \cos \frac{n\pi y}{y_1} \cos \frac{m\pi z}{z_1} e^{-\gamma x}$$

Hollow Rectangular Waveguide

- Step 8:

$$H_y = \frac{\gamma H_0}{k^2}\frac{n\pi}{y_1}\sin\frac{n\pi y}{y_1}\cos\frac{m\pi z}{z_1}e^{-\gamma x} \quad (61)$$

$$H_z = \frac{\gamma H_0}{k^2}\frac{m\pi}{z_1}\cos\frac{n\pi y}{y_1}\sin\frac{m\pi z}{z_1}e^{-\gamma x} \quad (62)$$

$$E_y = \frac{\gamma Z_{yz} H_0}{k^2}\frac{m\pi}{z_1}\cos\frac{n\pi y}{y_1}\sin\frac{m\pi z}{z_1}e^{-\gamma x} \quad (63)$$

$$E_z = -\frac{\gamma Z_{yz} H_0}{k^2}\frac{n\pi}{y_1}\sin\frac{n\pi y}{y_1}\cos\frac{m\pi z}{z_1}e^{-\gamma x} \quad (64)$$

If m=1, n=0: H_x, H_z, E_y

- For example: E_y has a sinusoidal variation across the guide (z-direction), being a max in the center and zero at the walls, and has no variation as a function of y.
- m, n → number of half-cycle variation wrt z and y respectively!

Hollow Rectangular Waveguide

- TE_{10}

$$E_x = 0 \qquad \text{TE mode requirement}$$

$$E_y = \frac{\gamma Z_{yz} H_0}{k^2} \frac{\pi}{z_1} \sin \frac{\pi z}{z_1} e^{-\gamma x}$$

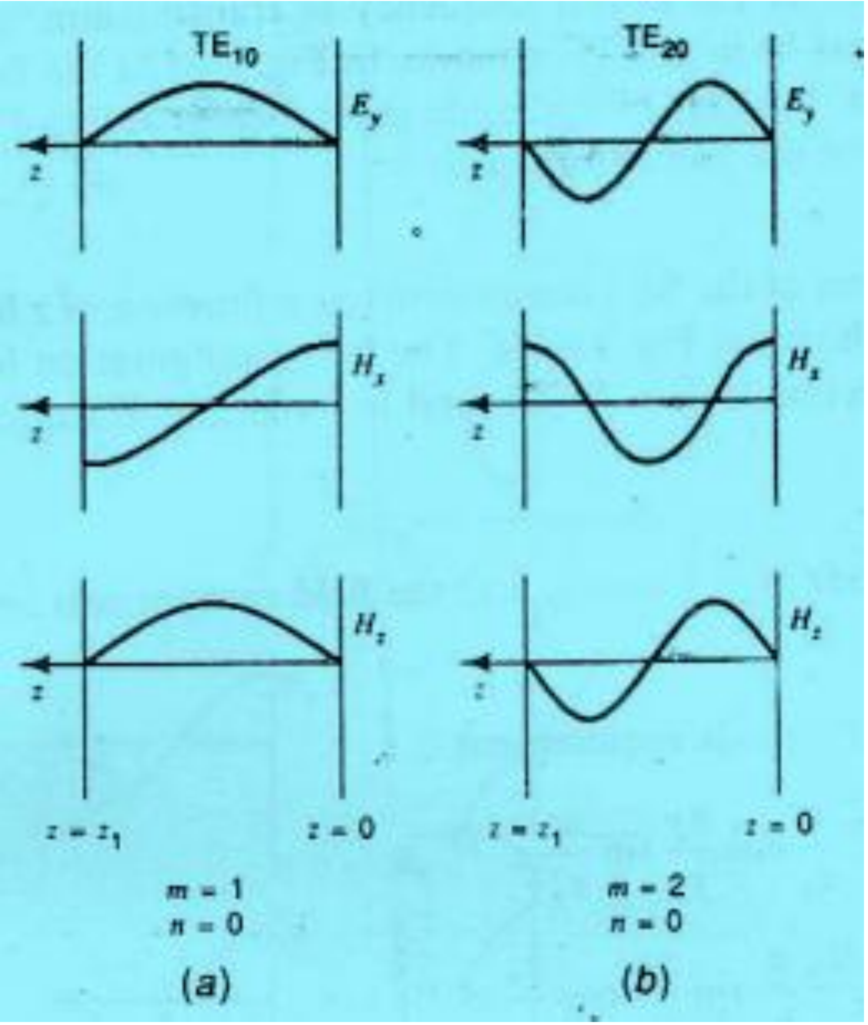
$$E_z = 0$$

$$H_x = H_0 \cos \frac{\pi z}{z_1} e^{-\gamma x} \qquad (65)$$

$$H_y = 0$$

$$H_z = \frac{\gamma H_0}{k^2} \frac{\pi}{z_1} \sin \frac{\pi z}{z_1} e^{-\gamma x}$$

Hollow Rectangular Waveguide



Hollow Rectangular Waveguide

- General Significance: $$\left(\frac{n\pi}{y_1}\right)^2 + \left(\frac{m\pi}{z_1}\right)^2 = k^2 \qquad (67)$$

$$k^2 = \gamma^2 - j\omega\mu(\sigma + j\omega\epsilon) \qquad (68)$$

For Lossless dielectric medium:

$$\gamma = \sqrt{\left(\frac{n\pi}{y_1}\right)^2 + \left(\frac{m\pi}{z_1}\right)^2 - \omega^2\mu\epsilon} \qquad (69)$$

1. At low frequencies, ω small, γ real, guide opaque, wave does not propagate
2. At an intermediate frequency, ω intermediate, $\gamma = 0$, transition condition (cutoff)
3. At high frequencies, ω large, γ imaginary, guide transparent, wave propagates