Queueing Theory

(Basics)

Contents

Queueing Theory

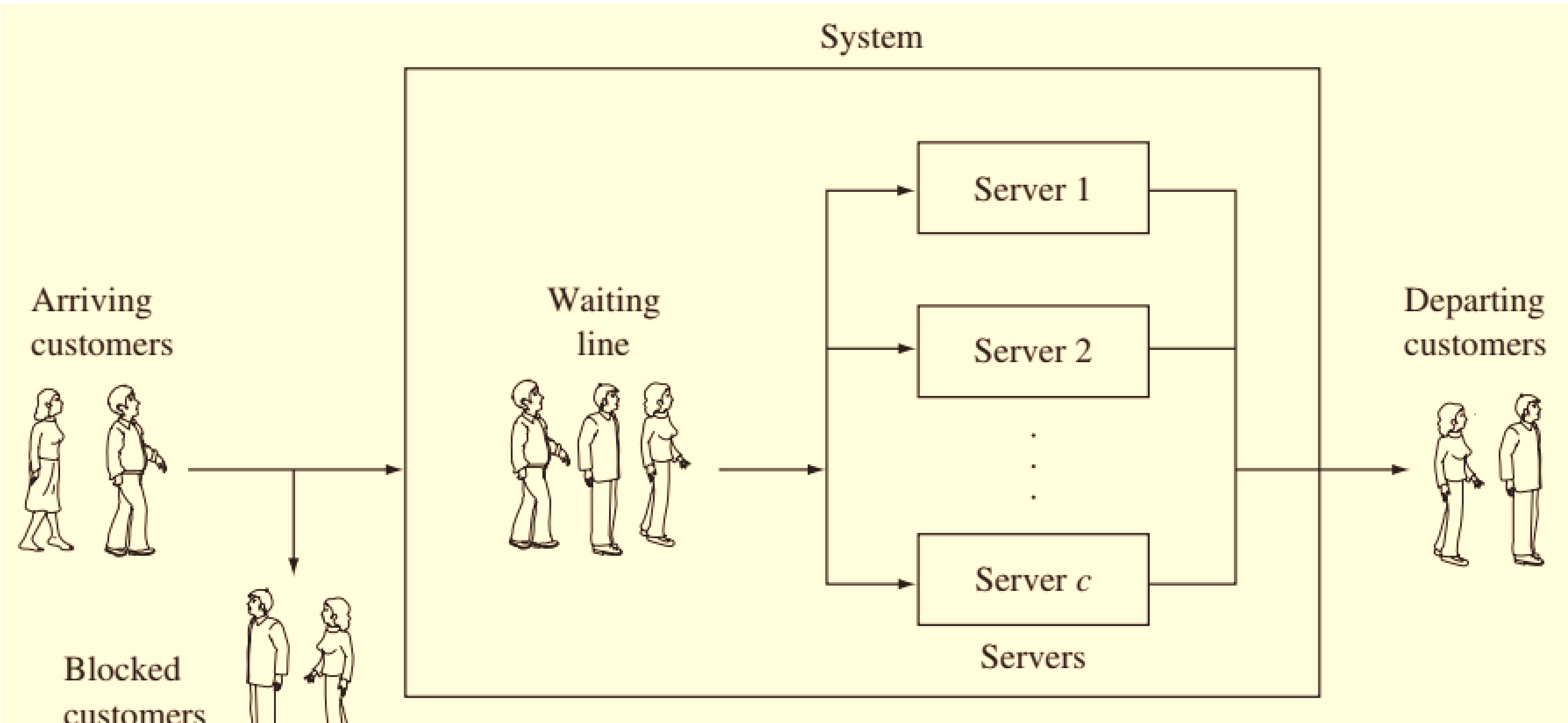
- **Queueing Theory:** Waiting lines and resource sharing

- **Queue Length:** Infinite and Finite

- **Population:** Finite and Infinite

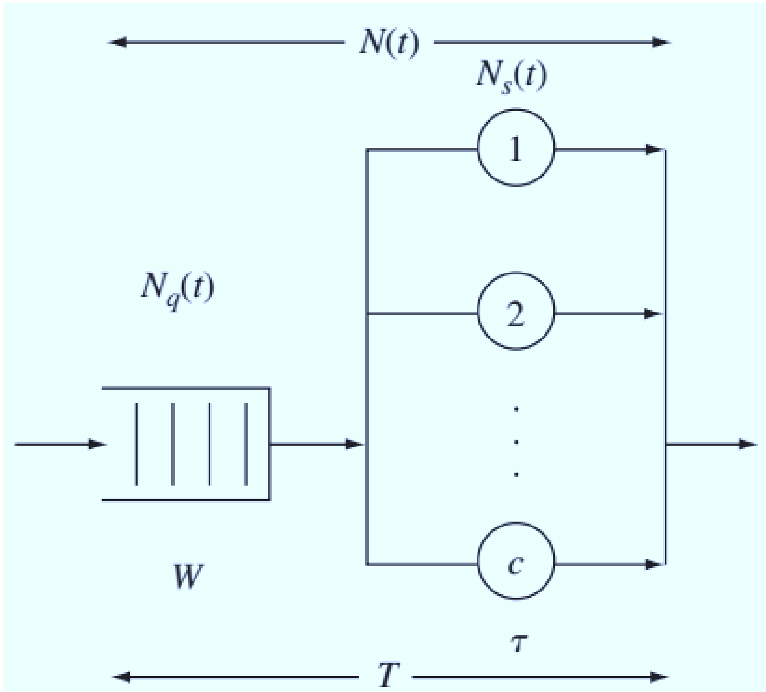
- **Queueing Behavior:** Balking, Reneging, and Jockeying

Elements of a Queueing System



(a)

Elements of a Queueing System Model N(t)



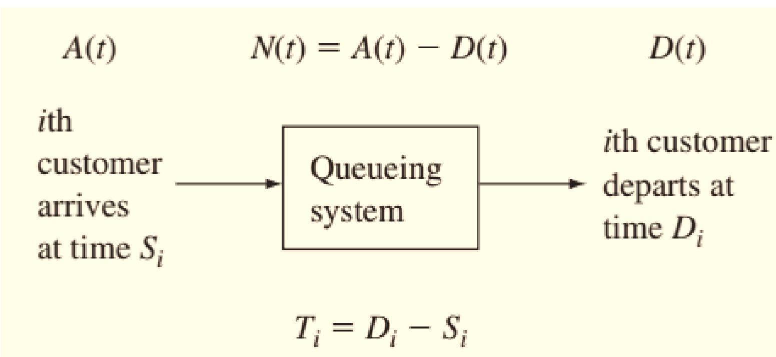
$$T_i = W_i + \tau_i.$$

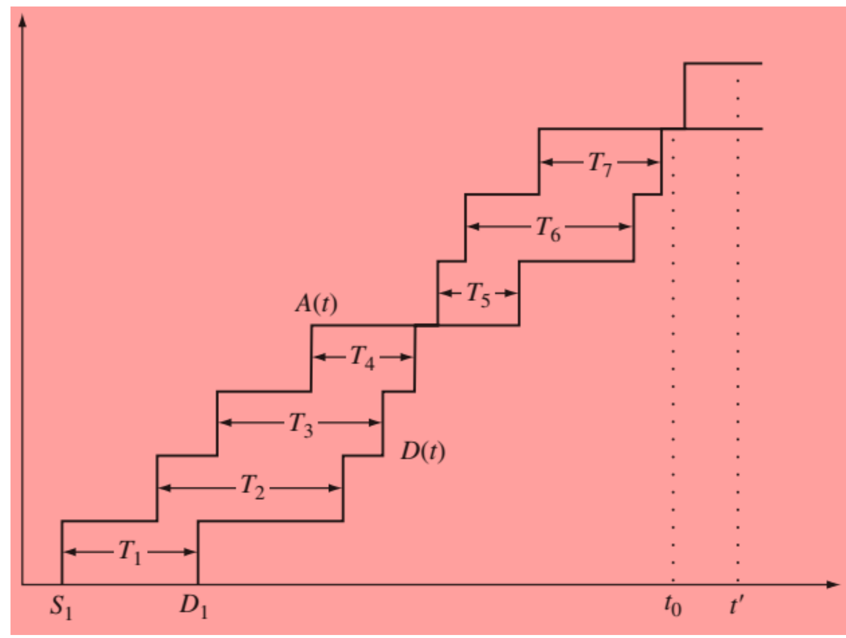
Queuing System: *a/b/m/K*

Little’s Formula

- Finds the average number of customers in steady state systems

$$E[N] = \lambda E[T].$$





$$\langle N \rangle_t = \frac{1}{t}\int_0^t N(t')\,dt'.$$

$$\langle N \rangle_t = \frac{1}{t}\sum_{i=1}^{A(t)} T_i.$$

Little's Formula

The average arrival rate up to time t is given by

$$\langle \lambda \rangle_t = \frac{A(t)}{t}.$$

$$\langle N \rangle_t = \langle \lambda \rangle_t \frac{1}{A(t)} \sum_{i=1}^{A(t)} T_i.$$

$$\langle T \rangle_t = \frac{1}{A(t)} \sum_{i=1}^{A(t)} T_i.$$

$$t \longrightarrow \infty,$$

$$\langle N \rangle_t = \langle \lambda \rangle_t \langle T \rangle_t$$

$$E[N] = \lambda E[T].$$

$$E[N_s] = \lambda E[\tau]$$

Example: Utilization

- Assume that the steady state probability that the system is empty:

$$p_0 = P[N(t) = 0]$$

- The system is busy with:

$$1 - p_0 = E[N_s] = \lambda E[\tau]$$

- The utilization of a single-server system

$$\rho = \lambda E[\tau].$$

- In general, the utilization of a c-server syetm

$$\rho = \frac{\lambda E[\tau]}{c}.$$

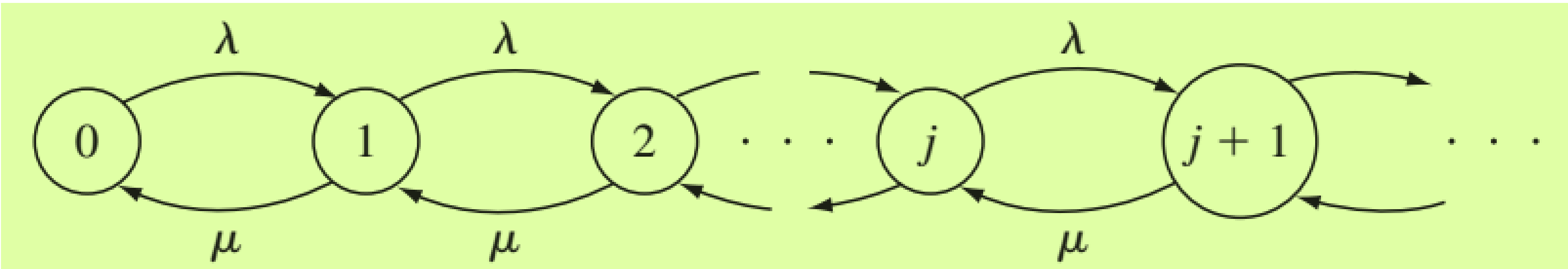
The M/M/1 Queue

- The steady state pmf of N(t), number of customers in the system
- The pdf of T, total delay in the system

- The Queue a Markov Chain

- Transition rates for N(t)
  - Probabilities of the various ways in which N(t) can change
  - Transition rate diagram

The M/M/1 Queue



- The global balance equations for the steady state probabilities

$$\lambda p_0 = \mu p_1$$

$$(\lambda + \mu) p_j = \lambda p_{j-1} + \mu p_{j+1} \qquad j = 1, 2, \ldots.$$

$$P[N(t) = j] = (1 - \rho)\rho^j \qquad j = 0, 1, 2, \ldots.$$

The Mean number of customers

$$E[N] = \sum_{j=0}^{\infty} j P[N(t) = j] = \frac{\rho}{1 - \rho},$$

The M/M/1 Queue

- The mean total customer delay

$$E[T] = \frac{E[N]}{\lambda} = \frac{\rho/\lambda}{1-\rho}$$
$$= \frac{1/\mu}{1-\rho} = \frac{E[\tau]}{1-\rho} = \frac{1}{\mu-\lambda}.$$

- The mean waiting time in the queue

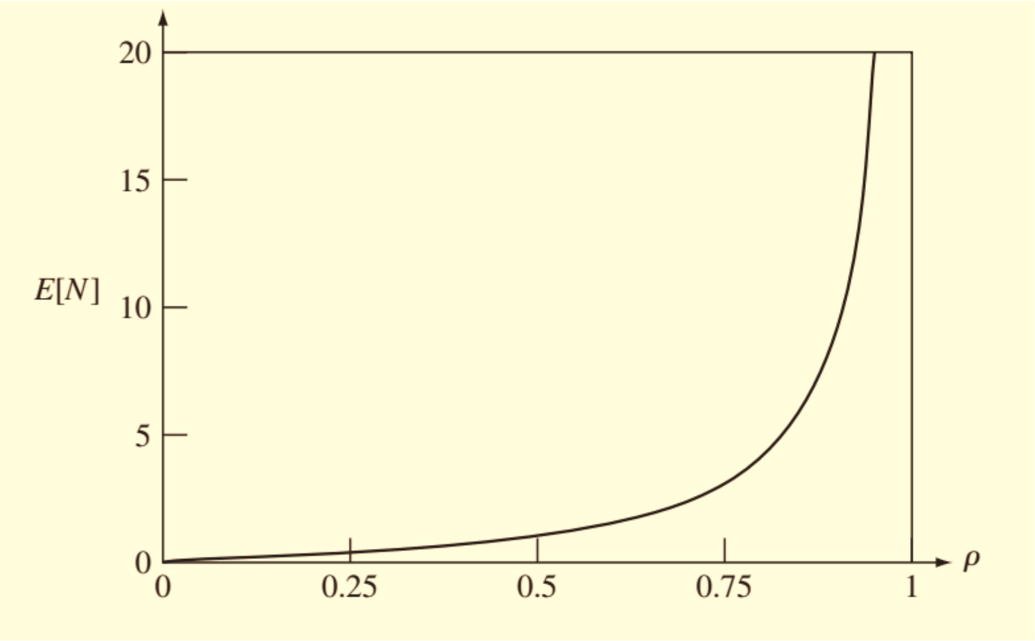
$$E[W] = E[T] - E[\tau]$$
$$= \frac{E[\tau]}{1-\rho} - E[\tau]$$
$$= \frac{\rho}{1-\rho}E[\tau].$$

- The mean number in the queue (Little's formula)

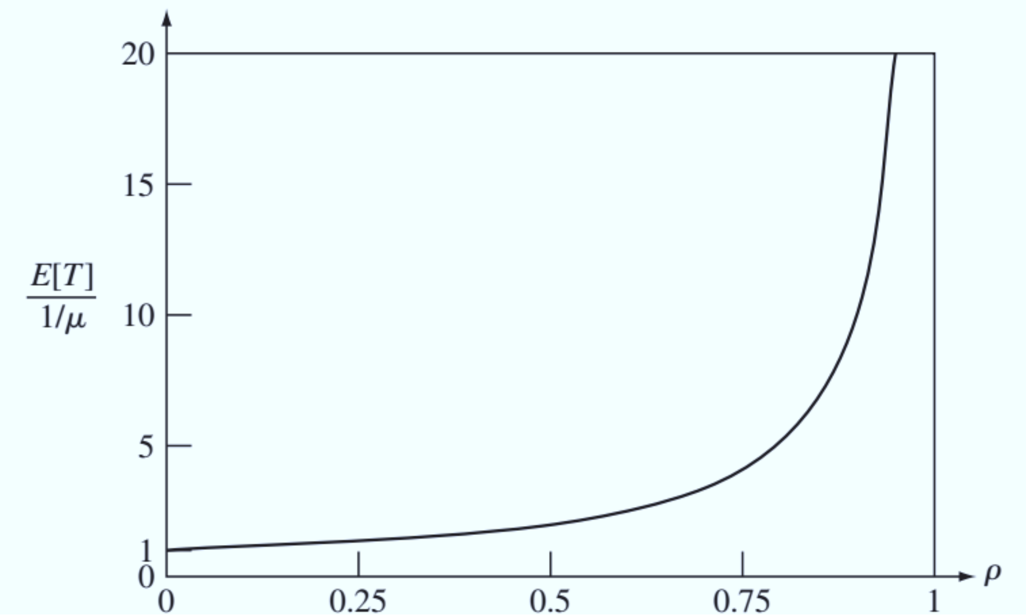
$$E[N_q] = \lambda E[W]$$
$$= \frac{\rho^2}{1-\rho}.$$

- The server utilization:

$$1 - p_0 = 1 - (1-\rho) = \rho = \frac{\lambda}{\mu}.$$



The M/M/1 Queue



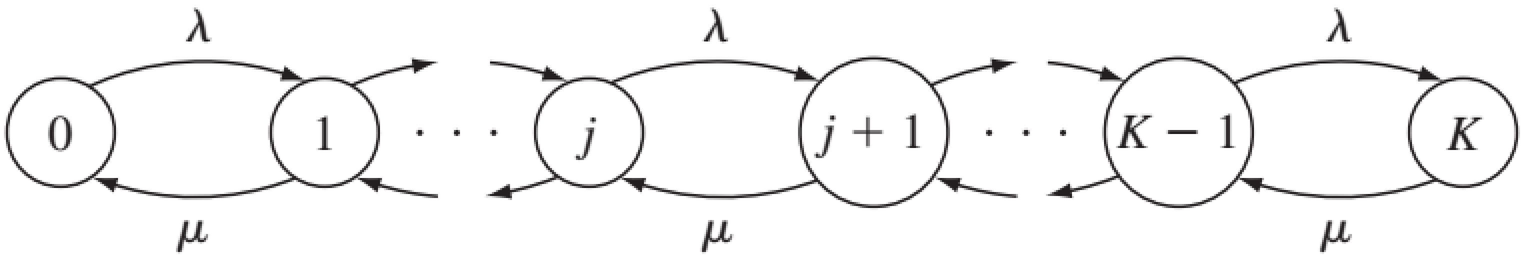
The M/M/1 Queue

- The pdf of T: $f_T(x) = (\mu - \lambda)e^{-(\mu-\lambda)x} \qquad x > 0.$

- The pdf of waiting time:

$$f_W(x) = (1-\rho)\delta(x) + \lambda(1-\rho)e^{-\mu(1-\rho)x} \qquad x > 0.$$

The M/M/1 System with Finite Capacity



The global balance equations are now

$$\lambda p_0 = \mu p_1$$

$$(\lambda + \mu) p_j = \lambda p_{j-1} + \mu p_{j+1} \qquad j = 1, 2, \ldots, K - 1$$

$$\mu p_K = \lambda p_{K-1}.$$

The M/M/1 System with Finite Capacity

- The steady state probabilities

$$P[N=j]=\frac{(1-\rho)\rho^j}{1-\rho^{K+1}} \qquad j=0,1,2,\ldots,K$$

- The mean number of customers

$$E[N]=\sum_{j=0}^{K} jP[N(t)=j]$$
$$=\begin{cases}\dfrac{\rho}{1-\rho}-\dfrac{(K+1)\rho^{K+1}}{1-\rho^{K+1}}\\ \dfrac{K}{2}\end{cases}$$

- The mean delay

$$E[T]=\frac{E[N]}{\lambda_a}=\frac{E[N]}{\lambda(1-p_K)}.$$

The M/M/1 System with Finite Capacity

- The traffic load offered to a system and the actual load carried by the system

- The offered load (or traffic intensity): a measure of demand on the system

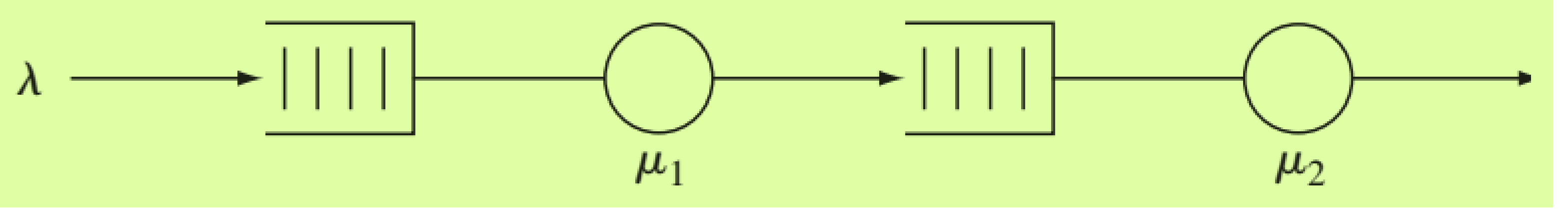
$$\lambda \frac{\text{customers}}{\text{second}} \times E[\tau] \frac{\text{seconds of service}}{\text{customer}}.$$

- The carried load is the actual demand met by the system

$$\lambda_a \frac{\text{customers}}{\text{second}} \times E[\tau] \frac{\text{seconds of service}}{\text{customer}}.$$

BURKE'S THEOREM

- DEPARTURES FROM M/M/C SYSTEMS

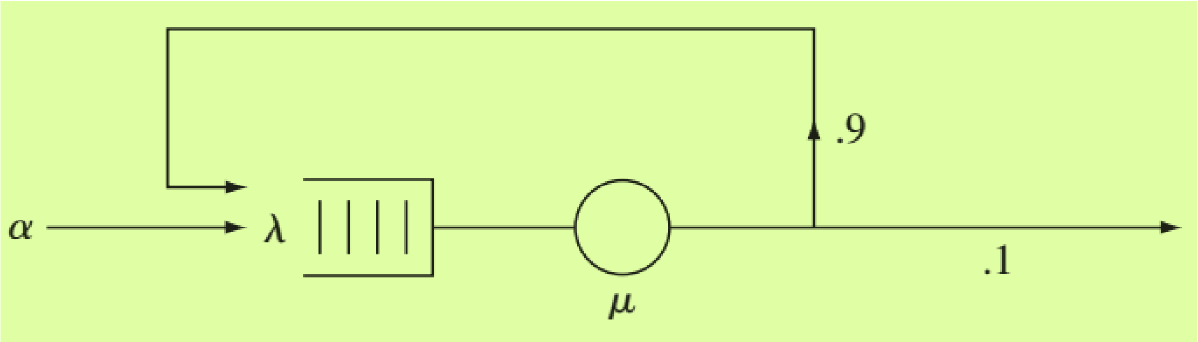


Consider an M/M/1, M/M/c, or M/M/∞ queueing system at steady state with arrival rate λ, then

1. The departure process is Poisson with rate λ.
2. At each time t, the number of customers in the system $N(t)$ is independent of the sequence of departure times prior to t.

Jackson's Theorem

- If a customer is allowed to visit a particular queue more than once



- JT:
  - The numbers of customers in the queues at time t are independent random variables.
  - The steady state probabilities of the individual queues are those of an M/M/c system.