Basic Concepts of Probability Theory

Random Processes

Specifying Random Experiments

- Examples of random experiments: tossing a coin, rolling a dice, the lifetime of a harddisk.
- Sample space: the set of all possible outcomes of a random experiment.
- Sample point: an element of the sample space S
- Examples: $S = \{H, T\}$

  $$S = \{1, 2, 3, 4, 5, 6\}$$

  $$S = \{t \mid 1 < t < 10\}$$

- Event: a subset of a sample space $A \subseteq S$

  $$A = \{H\}$$

  $$A = \{2, 4, 6\}$$

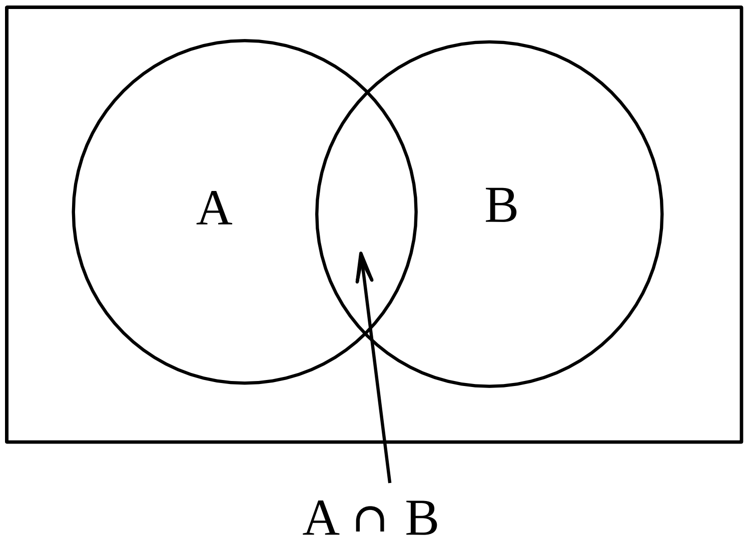
  $$A = \emptyset$$

The Axioms of Probability

- A probability measure is a set function $P(\cdot)$ that satisfies the following axioms.
  1. $P(A) \geq 0$
  2. $P(S) = 1$
  3. If $A \cap B = \emptyset$, then $P(A \cup B) = P(A) + P(B)$
  4. If $A_1, A_2, \cdots$ are events s.t $A_i \cap A_j = \emptyset$ for all $i \;!\!= j$ then $P\left(\cup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} P(A_k)$
- Corollary 1: $P(A^c) = 1 - P(A)$
- Corollary 2: $P(A) \leq 1 \quad \forall A$
- Corollary 3: $P(\emptyset) = 0$

The Axioms of Probability

- Corollary 4: If $A_1 \cdots A_n$ are mutually exclusive, i.e. $A_i \cap A_j = \emptyset, \ \forall i \neq j$ then $P(\cup_{k=1}^{n} A_k) = \sum_{k=1}^{n} P(A_k)$
- Corollary 5: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$



Union bound: $P(A \cup B) \leq P(A) + P(B)$

Counting Sample Points

- Permutation: n distinct objects, how many ways can we arrange them?

$$n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1 = n!$$

- Selection with order: select k objects from $n \geq k$ objects

$$n \cdot (n-1) \cdots (n-k+1) = \frac{n!}{(n-k)!} = {}_nP_k$$

- Combination (selection *without* order): select k objects from $n \geq k$ objects

$$C_k^n = \frac{n!}{(n-k)!k!} = \binom{n}{k}$$

Conditional Probability

- The conditional prob. of A given B has occurred is defined by :

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \quad P(B) > 0$$

Similarly,

$$P(B|A) = \frac{P(A \cap B)}{P(A)}, \quad P(A) > 0$$

So

$$P(A \cap B) = P(B)P(A|B) = P(A)P(B|A)$$

Example

A ball is selected from an urn containing two black balls, numbered 1 and 2, and two white balls, numbered 3 and 4. The number and color of the ball is noted, so the sample space is $\{(1, b), (2, b), (3, w), (4, w)\}$. Assuming that the four outcomes are equally likely, find $P[A \mid B]$ and $P[A \mid C]$, where A, B, and C are the following events:

$A = \{(1, b), (2, b)\}$, "black ball selected,"

$B = \{(2, b), (4, w)\}$, "even-numbered ball selected," and

$C = \{(3, w), (4, w)\}$, "number of ball is greater than 2."

Since $P[A \cap B] = P[(2, b)]$ and $P[A \cap C] = P[\varnothing] = 0$, Eq. (2.24) gives

$$P[A \mid B] = \frac{P[A \cap B]}{P[B]} = \frac{.25}{.5} = .5 = P[A]$$

$$P[A \mid C] = \frac{P[A \cap C]}{P[C]} = \frac{0}{.5} = 0 \neq P[A].$$

In the first case, knowledge of B did not alter the probability of A. In the second case, knowledge of C implied that A had not occurred.

Bayes' Rule

- If $B_1 \cdots B_n$ are mutually exclusive and $\cup B_i = S$, we call these sets a partition of S.
- Theorem of total probability

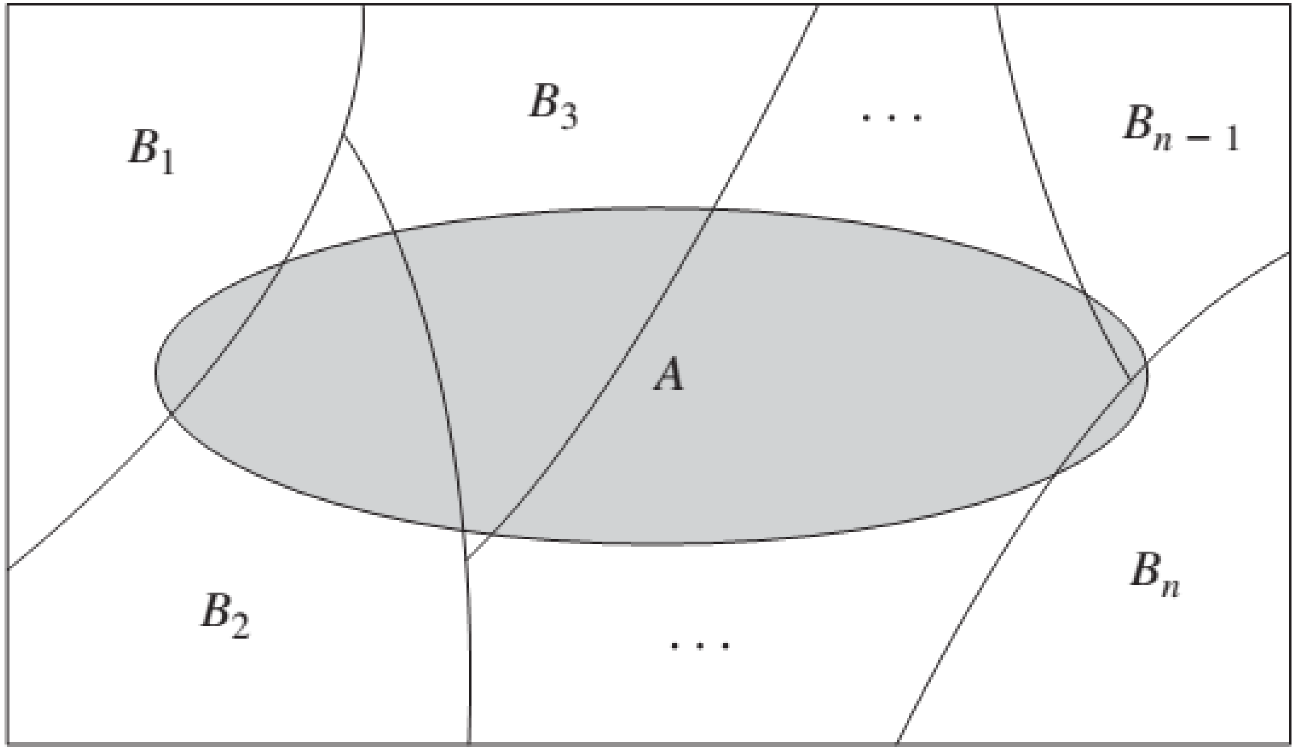
  For any event A, if $B_1, \cdots, B_n$ is a partition of S,

$$P(A) = P(A \cap B_1) + P(A \cap B_2) + \cdots + P(A \cap B_n)$$
$$= P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + \cdots + P(A|B_n)P(B_n)$$

- Bayes' Rule

  Let $B_1, \cdots, B_n$ be a partition of S, and $P(A) \geq 0$, then

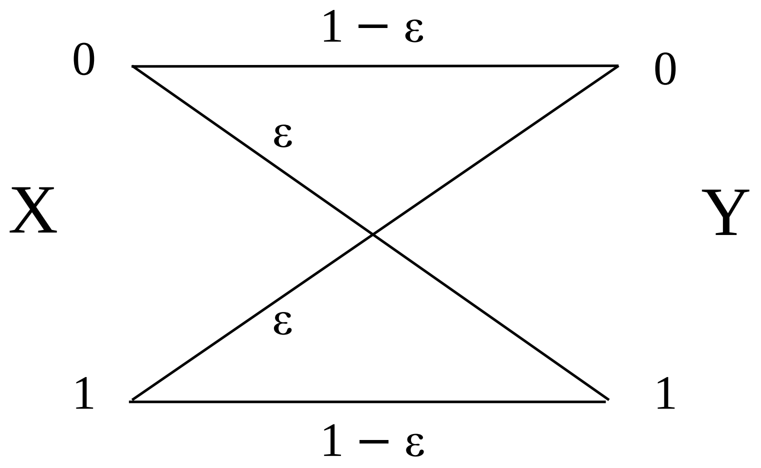
$$P(B_j|A) = \frac{P(A \cap B_j)}{P(A)} = \frac{P(A|B_j)P(B_j)}{\sum_{k=1}^{n} P(A|B_k)P(B_k)}$$



B1
B3
. . .
Bn − 1
A
B2
. . .
Bn

Bayes' Rule

- Example: binary communication system



$$P(X = 0) = P$$
$$P(X = 1) = 1 - P$$

1. Find $P(Y = 0)$
2. Find $P(X = 0 | Y = 0)$ and $P(X = 1 | Y = 0)$

Bayes' Rule

- Solution: let A be event $Y = 0$, B_0 be $X = 0$, B_1 be $X = 1$

$$P(Y = 0) = P(A) = P(A|B_0)P(B_0) + P(A|B_1)P(B_1)$$
$$= (1 - \varepsilon)P + \varepsilon(1 - P)$$

$$P(X = 0|Y = 0) = P(B_0|A) = \frac{P(A \cap B_0)}{P(A)}$$

$$= \frac{P(A|B_1)P(B_1)}{P(A)} = \frac{(1 - \varepsilon)P}{(1 - \varepsilon)P + \varepsilon(1 - P)}$$

Similarly:

$$P(X = 1|Y = 0) = \frac{\varepsilon(1 - P)}{(1 - \varepsilon)P + \varepsilon(1 - P)}$$

Bayes' Rule

- If $P(X=0|Y=0) > P(X=1|Y=0)$, we decide $X=0$, i.e., if

$$(1-\varepsilon)P > \varepsilon(1-P) \quad \Rightarrow \quad P > \varepsilon,$$

  we decide $X=0$.
- Similarly, if $P<\varepsilon$, we decide $X=1$.
- A special case: if $P=0.5$ and $\varepsilon<0.5$ then

$$Y=0 \Rightarrow \text{we decide } X=0$$
$$Y=1 \Rightarrow \text{we decide } X=1$$

Bayes' Rule

- Example: 2% of people have on type of blood disease. If a person has the disease and take a blood test, with 96% probability, the result is positive and with 4% probability, the result is negtive. If a person without the disease takes a blood test, then 94% negative and 6% positive. Find

$$P(\text{have disease}|\text{positive})$$

Bayes' Rule

- Solution: Let

  D: a person has the disease D^0: no disease

  B: blood test positive B^0: blood test negative

  Then $P(D) = 2\%$ and

  $P(B|D) = 0.96$ $P(B^0|D) = 0.04$

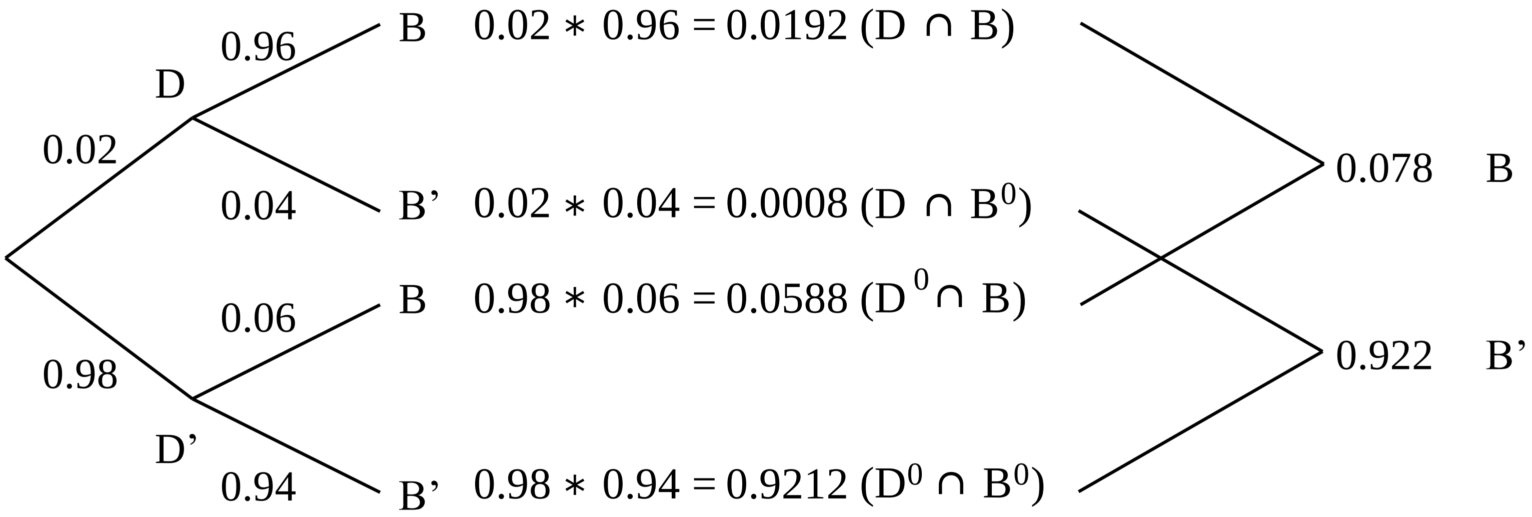
  $P(B|D^0) = 0.06$ $P(B^0|D^0) = 0.94$

  Apply Bayes Rule:

$$P(D|B) = 0.246 = 24.6\%$$

Bayes' Rule

- We can also use tree diagram.



$$P(D|B) = \frac{P(D \cap B)}{P(B)} = \frac{0.0192}{0.078} = 0.246$$

Bayes' Rule

- How to find $P(D|BB)$?

$$P(D|BB) = \frac{P(D \cap BB)}{P(BB)}$$

$$= \frac{0.02 * 0.96 * 0.96}{0.02 * 0.96 * 0.96 + 0.98 * 0.06 * 0.06}$$

$$= 83.9\%$$

Independence of Events

- A and B are independent iff

$$P(A \cap B) = P(A)P(B)$$

- If A and B are independent

$$\begin{aligned} P(A|B) &= \frac{P(A \cap B)}{P(B)} \\ &= \frac{P(A)P(B)}{P(B)} \\ &= P(A) \end{aligned}$$

Similarly

$$P(B|A) = P(B)$$

Independence of Events

- Events $A_1, A_2, \cdots, A_n$ are independent if
  $$P(A_{i_1} \cap A_{i_2} \cdots \cap A_{i_k}) = P(A_{i_1})P(A_{i_2}) \cdots P(A_{i_k})$$
  where $1 \le i_1 < i_2 \cdots < i_k \le n$
- A special case $P(A_1 \cap \cdots \cap A_n) = \prod_{i=1}^{n} P(A_n)$
- In general if events are not necessarily independent

$$P(A_1 \cap A_2 \cdots \cap A_n) = P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2) \cdots$$
$$\cdots P(A_n|A_1 \cap \cdots \cap A_{n-1})$$